5602   **Chapter 3: Preface to Grades Eight Through Twelve**

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5604   The standards for grades eight through twelve are organized differently from those

5605   for kindergarten through grade seven. (A complete description of this organization is

5606   provided on page 72, "Introduction to Grades Eight Through Twelve.") In grades

5607   eight through twelve, the mathematics studied is organized according to disciplines

5608   such as algebra and geometry. *Local educational agencies may choose to teach*

5609   *high school mathematics in a traditional sequence of courses (Algebra I, geometry,*

5610   *Algebra II, and so forth) or in an integrated fashion in which some content from each*

5611   *discipline is taught each year.*

5612   However mathematics courses are organized, the core content of these subjects

5613   must be covered by the end of the sequences of courses, and all academic

5614   standards for achievement must be the same. The core content and the emphasis

5615   areas are delineated in the discussions of the individual disciplines presented in this

5616   section.

5617   What follows in this preface is a discussion of key standards and discipline-level

5618   emphases for Algebra I, geometry, Algebra II, and probability and statistics. These

5619   same disciplines will be tested under the statewide Standardized Testing and

5620   Reporting (STAR) program, which will offer both traditional discipline-based versions

5621   and integrated versions of its test. The following section describes standards for the

5622   academic content by discipline, along with the areas of emphasis in each discipline;

5623   it is not an endorsement of a particular choice of structure for courses or a particular

5624   method of teaching the mathematical content. The additional advanced subjects of

5625   mathematics covered in the standards (linear algebra, advanced placement

5626   probability and statistics, and calculus) are not discussed in this section because

5627   many of these advanced subjects are not taught in every middle school or high

5628   school. Schools and districts may combine the subject matter of these various

5629   disciplines. Many combinations of these subjects are possible, and this framework

5630   does not prescribe a single instructional approach.

5631   By the eighth grade, students' mathematical sensitivity should be sharpened.

5632   Students should start perceiving logical subtleties and appreciating the need for

5633   sound mathematical arguments before making conclusions. As students progress in

5634   the study of mathematics, they learn to understand the meaning of logical

5635   implication; test general assertions; realize that one counterexample is enough to

5636   show that a general assertion is false; conceptually understand that the truth of a

5637   general assertion in a few cases does not allow the conclusion that it is true in all

5638   cases; distinguish between something being proven and a mere plausibility

5639   argument; and identify logical errors in chains of reasoning.

5640   From kindergarten through grade seven, these standards have impressed on the

5641   students the importance of logical reasoning in mathematics. Starting with grade

5642   eight, students should be ready for the basic message that logical reasoning is the

5643   underpinning of all mathematics. In other words, every assertion can be justified by

5644   logical deductions from previously known facts. Students should begin to learn to

5645   *prove* every statement they make. Every textbook or mathematics lesson should try

5646   to convey this message and to convey it well.

5647   [For information on problems from the Third International Study of Mathematics and

5648   Science (TIMSS), readers are referred to a resource kit, *Attaining Excellence: A TIMSS*

5649   *Resource Kit,* and to a Web site http://www.csteep.bc.edu/TIMSS1/pubs_main.html.]

5650   **Mathematical Proofs**

5651   A misapprehension in mathematics education is that proofs occur only in

5652   Euclidean geometry and that elsewhere one merely learns to *solve problems* and *do*

5653   *computations*. Problem solving and symbolic computations are nothing more than

5654   different manifestations of mathematical proofs. To illustrate this point, the following

5655   discussion shows how the usual computations leading to the solution of a simple

5656   linear equation are nothing but the steps of a well-disguised proof of a theorem.

5657   Consider the problem of solving this equation:

5658 $x - \tfrac{1}{4}(3x - 1) = 2x - 5$

5659   Multiply both sides by 4 to get:

5660 $4x - (3x - 1) = 8x - 20$

5661   Then simplify the left side to get:

5662 $x + 1 = 8x - 20$

5663   Transposing x from left to right yields:

5664 $1 = 7x - 20$

5665   One more transposition, and a division, gives the result $x = 3$.

5666   This would seem to be an entirely mechanical procedure that involves no proof

5667   because both the hypothesis and conclusion are hidden.

5668   Closer examination reveals that what is really being stated is a mathematical

5669   theorem:

5670   A number x satisfies $x - \tfrac{1}{4}(3x - 1) - 2x - 5$

5671   when and only when $x = 3$.

5672   That $x = 3$ satisfies the equation is easy to see. The less trivial part of the preceding

5673   theorem is the assertion that if a number x satisfies $x - \tfrac{1}{4}(3x - 1) - 2x - 5$, then x is

5674   necessarily equal to 3. A *proof* of this fact is presented next in a two-column format:

5675   1. $x - \tfrac{1}{4}(3x - 1) = 2x - 5$

5676   2. $4(x - \tfrac{1}{4}(3x - 1)) = 4(2x - 5)$

5677   3. $4x - 4(\tfrac{1}{4}(3x - 1)) = 4(2x) - 20$

5678   4. $4x - (4 \cdot \frac{1}{4})(3x - 1) = (4 \cdot 2)x - 20$

5679   5. $4x - (3x - 1) = 8x - 20$

5680   6. $4x + (-3x + 1) = 8x - 20$

5681   7. $(4x + (-3x)) + 1 = 8x - 20$

5682   8. $x + 1 = 8x - 20$

5683   9. $-x + (x + 1) = -x + (8x - 20)$

5684   10. $(-x + x) + 1 = (-x + 8x) - 20$

5685   11. $1 = 7x - 20$

5686   12. $1 + 20 = (7x - 20) + 20$

5687   13. $21 = 7x + [(-20) + 20]$

5688   14. $21 = 7x$

5689   15. $3 = x$

5690   16. $x = 3$

5691   1. Hypothesis

5692   2. $a = b$ implies $ca = cb$ for all numbers a, b, c.

5693   3. Distributive law

5694   4. Associative law for multiplication

5695   5. $1 \cdot a = a$ for all numbers a

5696   6. $-(a - b) = (-a + b)$ for all numbers a, b.

5697   7. Associative law for addition

5698   8. $4x + (-3x) = (4 + (-3))x$, by the distributive law

5699   9.   Equals added to equals are equal.

5700   10.   Associative law for addition

5701   11. $-x + 8x = (-1 + 8)x$, by the distributive law

5702   12.   Equals added to equals are equal.

5703   13.   Associative law for addition

5704   14. $-a + a = 0$ for all a; $b + 0 = b$ for all b.

5705   15.   Multiply (14) by $1/7$ and apply the associative law to $1/7(7x)$.

5706   16. $a = b$ implies that $b = a$ Q.E.D.

5707   The purpose of giving this proof is by no means to suggest that, in school

5708 mathematics, linear equations should ever be solved in this tedious fashion. Rather,

5709 the intention is to show that even certain standard procedures that students tend to

5710 take for granted are nevertheless mathematical proofs in disguise. Furthermore,

5711 without the realization that such a mathematical proof is lurking behind the well-

5712 known formalism of solving linear equations, an author of an algebra textbook or a

5713 teacher in a classroom would most likely emphasize the wrong points in the

5714 presentation of beginning algebra.

5715   The preceding proof clearly exposes the need for *generality* in the presentation of

5716 the associative laws and distributive law. In these standards these laws are taught

5717 starting with grade two, but it is probably difficult to convince students that such

5718 seemingly obvious statements deserve discussion. For example, if one has to

5719 believe that $3(5 + 11) = 3 \cdot 5 + 3 \cdot 11$, all one has to do is to expand both sides:

5720 clearly, $3 \cdot 16 = 15 + 33$ because both sides are equal to 48. However, one look at

5721 the deduction of step 3 from step 2 in the preceding mathematical demonstration

5722   would make it clear that the hands-on approach to the distributive law is useless in

5723   this situation. Begin with the right-hand side of the equation:

5724 $4(2x - 5) = 4(2x) - 4 \cdot 5$

5725   Here x is an *arbitrary* number, so we are not saying that

5726 $4(2 \cdot 17 - 5) = 4(2 \cdot 17) - 4 \cdot 5$

5727   or that

5728 $4(2 \cdot 172 - 5) = 4(2 \cdot 172) - 4 \cdot 5$

5729   Were that the case, the equality could again have been verified by expanding both

5730   expressions. Rather, the assertion is that, *although we do not know what number x*

5731   *is*, nevertheless it is true that $4(2x - 5) = 4(2x) - 4 \cdot 5$. There is no alternative except

5732   to justify this general statement by using a general rule: the distributive law. The

5733   same comment applies to the other applications of the associative laws and the

5734   distributive law in the preceding proof.

5735   It must be recognized that some proofs may not be accessible until the later

5736   grades, such as the reason for the formula of the circumference of a circle, $C = 2\pi r$.

5737   Nevertheless, *every* technique taught in mathematics is nothing but proofs in

5738   disguise. The validity of this statement can be revealed by considering a special

5739   case, such as this word problem for grade eight:

5740   *Jan had a bag of marbles. She gave one-half to James and then one-third of the*

5741   *marbles still in the bag to Pat. She then had 6 marbles left. How many marbles*

5742   *were in the bag to start with?* (TIMSS, gr. 8, N-16)

5743   The solution to the problem follows:

5744   Suppose Jan had n marbles to start with. If she gave one-half to James, then

5745   she had $n/2$ marbles left. According to the problem, she then gave one-third of

5746   what was left to Pat (i.e., she gave $(1/3) \cdot (1/2)n$ to Pat). Thus she gave $(1/6)n$

263

5747   marbles to Pat, and what she had left was $(1/2)n - (1/6)n = (1/3)n$. But the

5748   problem states that Jan had "6 marbles left." So $(1/3)n = 6$, and $n = 18$.

5749   Therefore, Jan had 18 marbles to begin with.

5750   The next step is to analyze in what sense the preceding solution masks a proof.

5751   First, the usual solution as presented previously can be broken into two distinct

5752   steps:

5753   1. *Setting up the equation:* If *n* is the number of marbles Jan had to begin with,

5754   then the given data imply:

5755 $$(n - (1/2)n) - (1/3)(n - (1/2)n) = 6$$

5756   2. *Solving the equation:* This step requires the proof of the following theorem:

5757   *n* satisfies the equation $(1/2)n - (1/3)(1/2)n = 6$ when and only when $n = 18$.

5758   Step 1 and step 2 exemplify the two components of mathematics in grades eight

5759   through twelve: teaching the skills needed to transcribe sometimes untidy raw data

5760   into mathematical terms and teaching the skills needed to draw precise logical

5761   conclusions from clearly stated hypotheses. Neither can be slighted.

5762   **Misconceptions in Mathematics Problems**

5763   It should be pointed out, however, that the built-in uncertainty and indeterminacy

5764   of step 1—which can lead to the setting up of several distinct equations and hence

5765   several distinct solutions—has led to the view of mathematics as an imprecise

5766   discipline in which a problem may have more than one correct answer. *This lack of*

5767   *understanding of the sharp distinction between step 1 and step 2 has had the*

5768   *deleterious effect of downgrading the importance of obtaining a single correct*

5769   *answer and jettisoning the inherent precision of mathematics.* As a result the rigor

5770   and precision needed for step 2 have been vigorously questioned. Such a

5771   misconception of mathematics would never have materialized had the process of

5772   transcription been better understood. This level of rigor and precision is embedded

5773   in the standards and is essential so that all students can develop mathematically to

5774   the level required in the *Mathematics Content Standards.*

5775   The following is an extreme example of the kind of misconception discussed

5776   earlier:

5777   *The 20 percent of California families with the lowest annual earnings pay an*

5778   *average of 14.1 percent in state and local taxes, and the middle 20 percent pay*

5779   *only 8.8 percent. What does that difference mean? Do you think it is fair? What*

5780   *additional questions do you have?*

5781   Any attempt to solve this problem would require a missing definition in

5782   mathematical terms of how to decide what is "fair," and consideration of much

5783   unspecified information about taxes and society. Since it is impossible to transcribe

5784   the problem as stated into mathematics, step 1 (setting up the equation) cannot be

5785   carried out, and so there can be no step 2 (solving the equation). This is therefore

5786   not a mathematical problem. Hence, the fact that it has no single correct answer can

5787   in no way lend credence to the assertion that mathematics is uncertain or imprecise.

5788   The preceding discussion explains that mathematical proofs are the underpinning

5789   of all of mathematics. Beginning with grade eight, students must deepen their

5790   understanding of the essential foundations for reasoning provided by mathematical

5791   proofs. It would be counterproductive to force every student to write a two-column

5792   proof at every turn, and it would be equally foolish to require all mathematics

5793   instructional materials to be as pedantic about giving such details as the two-column

5794   proof shown earlier in this preface. Nevertheless, the message that proofs underlie

5795   everything being taught should be clear in the instructional material and

5796   mathematical lessons taught in grades eight through twelve. In particular, all

5797   instructional materials—not just those for geometry, but especially those for algebra

5798   and trigonometry—should carefully present proofs of mathematical assertions when

5799   the situation calls for them. For example, an algebra textbook which asserts that a

5800   polynomial $p(x)$ satisfying $p(a) = 0$ for some number a must contain $x - a$ as a factor,

5801   but which does not offer a detailed proof beyond a few concrete examples for

5802   corroboration, is not presenting material compatible with the standards.

5803 **Chapter 3: Algebra I (Grades 8-12)**

5804

5805 In algebra, students learn to reason symbolically, and the complexity and types of

5806 equations and problems that they are able to solve increase dramatically as a

5807 consequence. The key content for the first course, Algebra I, involves

5808 understanding, writing, solving, and graphing linear and quadratic equations,

5809 including systems of two linear equations in two unknowns. Quadratic equations

5810 may be solved by factoring, completing the square, using graphs, or applying the

5811 quadratic formula. Students should also become comfortable with operations on

5812 monomial and polynomial expressions. They learn to solve problems employing all

5813 of these techniques, and they extend their mathematical reasoning in many

5814 important ways, including justifying steps in an algebraic procedure and checking

5815 algebraic arguments for validity.

5816 **Transition from Arithmetic to Algebra**

5817 Perhaps the fundamental difficulty for many students making the transition from

5818 arithmetic to algebra is their failure to recognize that the symbol x stands for a

5819 number. For example, the equation $3(2x - 5) + 4(x - 2) = 12$ simply means that a

5820 certain number x has the property that when the arithmetic operations $3(2x - 5) +$

5821 $4(x - 2)$ are performed on it as indicated, the result is 12. The problem is to find that

5822 number (solution). Teachers can emphasize this point by having students perform a

5823 series of arithmetic computations (using pen and paper) starting with $x = 1$, $x = 2$,

5824 $x = 3$, $x = 4$, and so forth, thereby getting -13, -3, 7, 17, and so forth. These

5825 computations show that none of 1, 2, 3, 4 can be that solution. Going from $x = 3$ to

5826 $x = 4$, the value of the expression changes from 7 to 17; therefore, it is natural to

5827 guess that the solution would be between 3 and 4. More experimentation eventually

5828 gives 3.5 as the solution.

5829   Working backwards, since $3(2(3.5) - 5) + 4((3.5) - 2) = 12$, one can apply the

5830   distributive law and commutative and associative laws to unwind the expression,

5831   intentionally not multiplying out 2(3.5), 4(3.5), and so forth, to get:

5832 $$3.5 = \frac{12 + 3(5) + 2(4)}{3(2) + 4}$$

5833   But this is exactly the principle of solving the equation $3(2x - 5) + 4(x - 2) = 12$ for

5834   the number x:

5835 $$x = \frac{12 + 3(5) + 2(4)}{3(2) + 4}$$

5836   One can bring closure to such a lesson by stressing the similarity between the

5837   handling of the algebraic equation and the earlier simple arithmetic operations.

5838   Basic Skills for Algebra I

5839   The first basic skills that must be learned in Algebra I are those that relate to

5840   understanding linear equations and solving systems of linear equations. In Algebra I

5841   the students are expected to solve only two linear equations in two unknowns, but

5842   this is a basic skill. The following six standards explain what is required:

5843   **4.0** Students simplify expressions before solving linear equations and inequalities

5844   in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.

5845   **5.0** Students solve multi-step problems, including word problems, involving linear

5846   equations and linear inequalities in one variable and provide justification for

5847   each step.

5848   **6.0** Students graph a linear equation and compute the x- and y- intercepts (e.g.,

5849   graph $2x + 6y = 4$). They are also able to sketch the region defined by linear

5850   inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

5851   **7.0** Students verify that a point lies on a line, given an equation of the line.

5852   Students are able to derive linear equations by using the point-slope formula.

5853   **9.0** Students solve a system of two linear equations in two variables algebraically

5854   and are able to interpret the answer graphically. Students are able to solve a

5855   system of two linear inequalities in two variables and to sketch the solution

5856   sets.

5857   **15.0** Students apply algebraic techniques to solve rate problems, work problems,

5858   and percent mixture problems.

5859   Each of these standards can be a source of difficulty for students, but they all

5860 reflect basic skills that must be understood so that students can advance to the next

5861 level in their understanding of mathematics. Moreover, modern applications of

5862 mathematics rely on solving systems of linear equations more than on any other

5863 single technique that students will learn in kindergarten through grade twelve

5864 mathematics. Consequently, it is essential that they learn these skills well.

5865 **Point-Slope Formula**

5866   Perhaps the most perplexing difficulty that students have is with Standard 7.0. It

5867 often seems very hard for them to understand this point. But it is one of the most

5868 critical skills in this section. In particular, the following idea must be clearly

5869 understood before the students can progress further: A point lies on a line given by,

5870 for example, the equation $y = 7x + 3$ if and only if the coordinates of that point (a, b)

5871 satisfy the equation when x is replaced with a and y with b. One way of explaining

5872 this idea is to emphasize that the graph of the equation $y = 7x + 3$ is precisely the set

5873 of points (a, b) for which replacing x by a and y by b gives a true statement. (For

5874 example, $(3, 2)$ is not on the graph because replacing x with 3 and y with 2 gives the

5875 statement $2 = 23$, which is not true.) Thus, the graph consists of all points of the form

5876 $(a, 7a + 3)$. It also follows from these considerations that the root r of the linear

5877   polynomial $7x + 3$ is the *x*-intercept of the graph of $y = 7x + 3$ because $(r, 0)$ is on the

5878   graph.

5879   An additional comment about Standard 7.0 is that, although it singles out the

5880   point-slope formula, it is understood that students also have to know how to write the

5881   equation of a line when two of its points are given. However, the fact that the slope

5882   of a line is the same regardless of which pair of points on the line are used for its

5883   definition depends on the considerations of similar triangles. (This fact is first

5884   mentioned in Algebra and Functions Standard 3.3 for grade 7.) This small gap in the

5885   logical development should be made clear to students, with the added assurance

5886   that they will learn the concept in geometry. The same comment applies also to the

5887   fact that two non-vertical lines are perpendicular if and only if the product of their

5888   slopes is -1 (Standard 8.0).

5889   **Quadratic Equations**

5890   The next basic topic is the development of an understanding of the structure of

5891   quadratic equations. Here, one repeats the considerations involved in linear

5892   equations, such as graphing and understanding what it means for a point (x, y) to be

5893   on the graph. In particular, the graphical interpretation of finding the zeros of a

5894   quadratic equation by identifying the *x*-intercepts with the graph is very important

5895   and, as was the case with linear equations, is also a source of serious difficulty.

5896   Equally important is the recognition that if a, b are the roots of a quadratic

5897   polynomial, then up to a multiplicative constant, it is equal to $(x - a)(x - b)$.

5898   When the discriminant of a quadratic polynomial is negative, the quadratic formula

5899   yields no information at this point because students have not yet been introduced to

5900   complex numbers. This deficiency will be remedied in Algebra II. The following

5901   standards show which skills students in a first-year algebra course need for solving

5902   quadratic equations:

5903   **14.0** Students solve a quadratic equation by factoring or completing the square.

5904   **19.0** Students know the quadratic formula and are familiar with its proof by

5905   completing the square.

5906   **20.0** Students use the quadratic formula to find the roots of a second-degree

5907   polynomial and to solve quadratic equations.

5908   **21.0** Students graph quadratic functions and know that their roots are the *x*-

5909   intercepts.

5910   **23.0** Students apply quadratic equations to physical problems, such as the motion

5911   of an object under the force of gravity.

5912   **Additional Comments**

5913   Students should be carefully guided through the solving of word problems by

5914   using symbolic notations. Many students may be so overwhelmed by the symbolic

5915   notation that they start to manipulate symbols carelessly, and word problems

5916   become incomprehensible. Teachers and publishers need to be sensitive to this

5917   difficulty. In addition to Standard 15.0, cited previously, the other relevant standards

5918   for solving word problems using symbolic notations are:

5919   **10.0** Students add, subtract, multiply and divide monomials and polynomials.

5920   Students solve multistep problems, including word problems, by using these

5921   techniques.

5922   **13.0** Students add, subtract, multiply, and divide rational expressions and

5923   functions. Students solve both computationally and conceptually challenging

5924   problems by using these techniques.

5925   Among the word problems of this level, those involving *direct* and *inverse*

5926   proportions occupy a prominent place. These concepts, which are often mired in the

5927   language of "proportional thinking," need clarification. A quantity *P* is said to be

5928   *proportional to* another quantity *Q* if the quotient *P*/*Q* is a fixed constant *k*. This *k* is

5929 then called the *constant of proportionality*. Students should be made aware that this

5930 is a mathematical definition, and there is no need to look for linguistic subtleties

5931 concerning the phrase "to be proportional to." Similarly, *P* is said to be *inversely*

5932 *proportional* to *Q* if the product *PQ* is equal to a fixed nonzero constant *h*.

5933   In Standard 13.0 the emphasis should be on *formal* rational expressions instead of

5934 on rational *functions*. Many of these formal techniques will become increasingly

5935 important in Algebra II and trigonometry. The rules of exponents, for example, are

5936 fundamental to an understanding of the exponential and logarithmic functions. Many

5937 students fail to cope with the latter topics because their understanding of the rules of

5938 (fractional) exponents is weak. The skills in the following standards need to be

5939 emphasized in a first-year algebra course:

5940   **2.0** Students understand and use such operations as taking the opposite, finding

5941   the reciprocal, taking a root, and raising to a fractional power. They

5942   understand and use the rules of exponents.

5943   **12.0** Students simplify fractions with polynomials in the numerator and

5944   denominator by factoring both and reducing them to the lowest terms.

5945   The gist of Standards 16.0 through 18.0 is to introduce students to a precise

5946 concept of functions in the language of ordered pairs. Introducing this concept needs

5947 to be done carefully because students at this stage of their mathematical

5948 development may not be ready for this level of abstraction. However, during a first-

5949 year algebra course is the stage at which students should see and use the functional

5950 notation *f*(*x*) for the first time.

5951   In Standard 24.0 students begin to learn simple logical arguments in algebra.

5952 They can be taught the proof that square roots of prime numbers are never rational,

5953 thereby solidifying to a certain extent their understanding of rational and irrational

5954 numbers (grade seven, Number Sense Standard 1.4). In Standard 3.0 students are

5955   taught to solve equations and inequalities involving absolute values, but it is not

5956   necessary to introduce the interval notation [*a*, *b*], (*a*, *b*), [*a*, *b*), and so forth at this

5957   point. However, they should be introduced to the set notation {*a*, *b*, *c*, . . .} and {*x*:*x*

5958   satisfies property *P*} and to the empty set ϕ in, for example, Standard 17.0. Finally,

5959   students should become familiar with the terminology "solution set" of Standard

5960   9.0—meaning the set of all solutions.

5961   **Chapter 3: Geometry (Grades 8-12)**

5962

5963   The main purpose of the geometry curriculum is to develop geometric skills and

5964   concepts and the ability to construct formal logical arguments and proofs in a

5965   geometric setting. Although the curriculum is weighted heavily in favor of plane

5966   (synthetic) Euclidean geometry, there is room for placing special emphasis on

5967   coordinated geometry and its transformations.

5968   The first standards introduce students to the basic nature of logical reasoning in

5969   mathematics:

5970   **1.0** Students demonstrate understanding by identifying and giving examples of

5971   undefined terms, axioms, theorems, and inductive and deductive reasoning.

5972   **3.0** Students construct and judge the validity of a logical argument and give

5973   counterexamples to disprove a statement.

5974   Starting with undefined terms and axioms, students learn to establish the validity

5975   of other assertions through logical deductions; that is, they learn to prove theorems.

5976   This is their first encounter with an axiomatic system, and experience shows that

5977   they do not easily adjust to the demand of total precision needed for the task. In

5978   general, it is important to impress on students from the beginning that the main point

5979   of a proof is the mathematical correctness of the argument, not the literary polish of

5980   the writing or the adherence to a particular proof format.

5981   **Inductive Reasoning**

5982   Standard 1.0 also calls for an understanding of inductive reasoning. Students are

5983   expected not only to recognize inductive reasoning in a formal sense but also to

5984   demonstrate how to put it to use. To this end students should be encouraged to

5985   draw many pictures to develop a geometric sense and to amass a wealth of

5986   geometric data in the process. Many students—including high-achieving ones—

5987   complete a course in geometry with so little geometric intuition that, given three

5988   noncollinear points, they cannot even begin to visualize what the circumcircle of

5989   these points must be like. One way to develop this geometric sense is to have the

5990   students become familiar with the basic straightedge-compass constructions, as

5991   illustrated in the following standard:

5992   **16.0** Students perform basic constructions with a straightedge and compass, such

5993   as angle bisectors, perpendicular bisectors, and the line parallel to a given

5994   line through a point off the line.

5995   It would be desirable to introduce students to these constructions early in the

5996   course and leave the proofs of their validity to the appropriate place of the logical

5997   development later.

5998   **Geometric Proofs**

5999   The subject then turns to geometric proofs in earnest. The foundational results of

6000   plane geometry are embodied in the following standards:

6001   **2.0** Students write geometric proofs, including proofs by contradiction.

6002   **4.0** Students prove basic theorems involving congruence and similarity.

6003   **7.0** Students prove and use theorems involving the properties of parallel lines cut

6004   by a transversal, the properties of quadrilaterals, and the properties of circles.

6005   **12.0** Students find and use measures of sides and of interior and exterior angles

6006   of triangles and polygons to classify figures and solve problems.

6007   **21.0** Students prove and solve problems regarding relationships among chords,

6008   secants, tangents, inscribed angles, and inscribed and circumscribed

6009   polygons of circles.

6010   It has become customary in high school geometry textbooks to start with axioms

6011   that incorporate real numbers. Although doing geometric proofs with real numbers

6012   runs counter to the spirit of Euclid, this approach is a good mathematical

6013   compromise in the context of school mathematics. However, the parallel postulate

6014   occupies a special place in geometry and should be clearly stated in the traditional

6015   form: Through a point not on a given line *L*, there is exactly one line parallel to *L*.

6016   Because this postulate played a fundamental role in the development of

6017   mathematics up to the nineteenth century, the significance of the postulate should

6018   be discussed. And because there always exists at least one parallel line through a

6019   point to a given line, the import of this postulate lies in the uniqueness of the parallel

6020   line. A discussion of this postulate provides a natural context to show students the

6021   key concept of uniqueness in mathematics—a concept that experience indicates

6022   students usually find difficult.

6023   It is also recommended that the topics of circles and similarity be taught as early

6024   as possible. Once those topics have been presented, the course enters a new phase

6025   not only because of the interesting theorems that can now be proved but also

6026   because the concept of similarity expands the applications of algebra to geometry.

6027   These applications might include determining one side of a regular decagon on the

6028   unit circle through the use of the quadratic formula as well as the applications of

6029   geometry to practical problems.

6030   It is often not realized that theorems for circles can be introduced very early in a

6031   geometry course. For instance, the remarkable theorem that inscribed angles on a

6032   circle which intercept equal arcs must be equal can in fact be presented within three

6033   weeks after the introduction of axioms. All it takes is to prove the following two

6034   theorems:

6035   1.   Base angles of isosceles triangles are equal.

6036   2. The exterior angle of a triangle equals the sum of opposite interior angles.

6037   At this point it is necessary to deal with one of the controversies in mathematics

6038   education concerning the format of proofs. It has been argued that the traditional

6039   two-column format is stultifying for students and that the format for proofs in the

6040   mathematics literature is always paragraph proofs. While the latter observation is

6041   true, teachers should be aware that a large part of the reason for using paragraph

6042   proofs is the expense of typesetting more elaborate formats, not that paragraph

6043   proofs are intrinsically better or clearer. In fact, neither of these claims of superiority

6044   for paragraph proofs is actually valid. Furthermore, it appears that for beginners to

6045   learn the precision of argument needed, the two-column format is best. After the

6046   students have shown a mastery of the basic logical skills, it would be appropriate to

6047   relax the requirements on form. *But the teacher should never relax the requirement*

6048   *that all arguments presented by the students be precise and correct.*

6049   **Pythagorean Theorem**

6050   One of the high points of elementary mathematics, in fact of all of mathematics, is

6051   the Pythagorean theorem:

6052   **14.0** Students prove the Pythagorean theorem.

6053   This theorem can be proved initially by using similar triangles formed by the

6054   altitude on the hypotenuse of a right triangle. Once the concept of area is introduced

6055   (Standard 8.0), students can prove the Pythagorean theorem in at least two more

6056   ways by using the familiar picture of four congruent right triangles with legs a and b

6057   nestled inside a square of side $a + b$.

6058   **8.0** Students know, derive, and solve problems involving the perimeter,

6059   circumference, area, volume, lateral area, and surface area of common

6060   geometric figures.

6061   **10.0** Students compute areas of polygons, including rectangles, scalene triangles,

6062   equilateral triangles, rhombi, parallelograms, and trapezoids.

6063   For rectilinear figures in the plane, the concept of area is simple because

6064   everything reduces to a union of triangles. However, the course must deal with

6065   circles, and here limits must be used and the number π defined. The concept of limit

6066   can be employed intuitively without proofs. If the area or length of a circle is defined

6067   as the limit of approximating, inscribing, or circumscribing regular polygons, then π is

6068   either the area of a disk of unit radius or the ratio of circumference to diameter, and

6069   heuristic arguments (see the glossary) for the equivalence of these two definitions

6070   would be given.

6071   The concept of volume, in contrast with that of area, is not simple even for

6072   polyhedra and should be touched on only lightly and intuitively. However, the

6073   formulas for volumes and surface areas of prisms, pyramids, cylinders, cones, and

6074   spheres (Standard 9.0) should be memorized.

6075   An important aspect of teaching three-dimensional geometry is to cultivate

6076   students' spatial intuition. Most students find spatial visualization difficult, which is all

6077   the more reason to make the teaching of this topic a high priority.

6078   The basic mensuration formulas for area and volume are among the main

6079   applications of geometry. However, the Pythagorean theorem and the concept of

6080   similarity give rise to even more applications through the introduction of

6081   trigonometric functions. The basic trigonometric functions in the following standards

6082   should be presented in a geometry course:

6083   **18.0** Students know the definitions of the basic trigonometric functions defined by

6084   the angles of a right triangle. They also know and are able to use elementary

6085   relationships between them. For example, $\tan(x) = \sin(x)/\cos(x)$, $(\sin(x))^2 +$

6086 $(\cos(x))^2 = 1$.

6087   **19.0** Students use trigonometric functions to solve for an unknown length of a side

6088   of a right triangle, given an angle and a length of a side.

6089   Finally, the Pythagorean theorem leads naturally to the introduction of rectangular

6090   coordinates and coordinate geometry in general. A significant portion of the

6091   curriculum can be devoted to the teaching of topics embodied in the next two

6092   standards:

6093   **17.0** Students prove theorems by using coordinate geometry, including the

6094   midpoint of a line segment, the distance formula, and various forms of

6095   equations of lines and circles.

6096   **22.0** Students know the effect of rigid motions on figures in the coordinate plane

6097   and space, including rotations, translations, and reflections.

6098   **The Connection Between Algebra and Geometry**

6099   These standards lead students to the next level of sophistication: an algebraic and

6100   transformation-oriented approach to geometry. Students begin to see how algebraic

6101   concepts add a new dimension to the understanding of geometry and, conversely,

6102   how geometry gives substance to algebra. Thus straight lines are no longer merely

6103   simple geometric objects; they are also the graphs of linear equations. Conversely,

6104   solving simultaneous linear equations now becomes finding the point of intersection

6105   of straight lines. Another example is the interpretation of the geometric concept of

6106   congruence in the Euclidean plane as a correspondence under an isometry of the

6107   coordinate plane. Concrete examples of isometries are studied: rotations,

6108   reflections, and translations. It is strongly suggested that the discussion be rounded

6109   off with the proof of the structure theorem: Every isometry of the coordinate plane is

6110   a translation or the composition of a translation and a rotation or the composition of

6111   a translation, a rotation, and a reflection.

6112   *Special attention should be given to the fact that a gap in Algebra I must be filled*

6113   *here.* Standards 7.0 and 8.0 of Algebra I assert that:

6114

6115   1. The graph of a linear equation is a straight line.

6116   2. Two straight lines are perpendicular if and only if their slopes have a product

6117   of −1.

6118   These facts should now be proved.

6119   **Additional Comments and Cautionary Notes**

6120   This section provides further comments and cautions in presenting the material in

6121   geometry courses.

6122   ***Introduction to Proofs.*** An important point to make to students concerning proofs

6123   is that while the written proofs presented in class should serve as models for

6124   exposition, *they should in no way be a model of how proofs are discovered*. The

6125   perfection of the finished product can easily mislead students into thinking that they

6126   must likewise arrive at their proofs with the same apparent ease. Teachers need to

6127   make clear to their students that the actual thought process is usually full of false

6128   starts and that there are many zigzags between promising leads and dead ends.

6129   Only trial and error can lead to a correct proof.

6130   This awareness of the nature of solving mathematical problems might lead to a

6131   deemphasis of the rigid requirements on the writing of two-column proofs in some

6132   classrooms.

6133   ***Students' perceptions of proofs.*** The first part of the course sets the tone for

6134   students' perceptions of proofs. With this in mind, it is advisable to discuss, mostly

6135   *without proofs*, those first consequences of the axioms that are needed for later

6136   work. A few proofs should be given for illustrative purposes; for example, the

6137　equality of vertical angles or the equality of the base angles of an isosceles triangle

6138　and its converse. There are two reasons for the recommendation to begin with only

6139　a few proofs. The foremost is that a complete logical development is neither possible

6140　nor desirable. This has to do with the intrinsic complexity of the structure of

6141　Euclidean geometry (see Greenberg 1993, 1–146). A second reason is the usual

6142　misconception that such elementary proofs are easy for beginners. Working on the

6143　level of axioms is actually more difficult for beginners than working with the theorems

6144　that come a little later in the logical development. This difficulty occurs because, on

6145　the one hand, working with axioms requires a heavy reliance on formal logic without

6146　recourse to intuition—in fact often *in spite of* one's intuition. On the other hand,

6147　working on the level of axioms does not usually have a clear direction or goal, and it

6148　is difficult to convince students to learn something without a clearly stated goal. If

6149　one so desires, students can always be made to go back to prove the elementary

6150　theorems *after* they have already developed a firm grasp of proof techniques.

6151　　***Structured work with proofs.*** Students' first attempts at proofs need to be

6152　structured with care. At the beginning of the development of this skill, instead of

6153　asking students to do *many* trivial proofs after showing them the proofs of two or

6154　three easy theorems, it might be a good strategy to proceed as follows:

6155　1.　As early as possible, the students might be shown a generous number of proofs

6156　　　of substantive theorems so that they can gain an understanding of what a proof

6157　　　is before they write any proofs themselves.

6158　2.　As a prelude to constructing proofs themselves, the students might provide

6159　　　reasons for some of the steps in the sample (substantive) proofs instead of

6160　　　constructing extremely easy proofs on their own.

6161　3.　After an extended exposure to nontrivial proofs, students might be asked to give

6162　　　proofs of simple corollaries of substantive theorems.

6163   The reason for steps 2 and 3 is to make students, from the beginning, associate

6164   proofs with real mathematics rather than perform a formal ritual. This goal can be

6165   accomplished with the use of *local axiomatics;* that is, if the proof of a theorem

6166   makes use of facts not previously proved, let these facts be stated clearly before the

6167   proof. These facts need not be previously proven but should ideally be sufficiently

6168   plausible even without a proof. Extensive use of local axiomatics would make

6169   possible, sufficiently early in the course, the presentation of interesting but perhaps

6170   advanced theorems. *In Appendix D, "Resource for Secondary School Teachers:*

6171   *Circumcenter, Orthocenter, and Centroid," the ideas in steps 2 and 3 are put to use*

6172   *to demonstrate how they might work.*

6173   ***Development of geometric intuition.*** The following geometric constructions are

6174   recommended to develop students' geometric intuition. (In this context *construction*

6175   means "construction with straightedge and compass.") It is understood that all of

6176   them will be proved at some time during the course of study. The constructions that

6177   students should be able to do are:

6178   • Bisecting an angle

6179   • Constructing the perpendicular bisector of a line segment

6180   • Constructing the perpendicular to a line from a point on the line and from a point

6181   not on the line

6182   • Duplicating a given angle

6183   • Constructing the parallel to a line through a point not on the line

6184   • Constructing the circumcircle of a triangle

6185   • Dividing a line segment into n equal parts

6186   • Constructing the tangent to a circle from a point on the circle

6187   • Constructing the tangents to a circle from a point not on the circle

6188   • Locating the center of a given circle

6189   • Constructing a regular *n*-gon on a given circle for *n* = 3, 4, 5, 6

6190   ***Use of technology.*** This is the place to add a word about the use of technology.

6191   The availability of good computer software makes the accurate drawing of geometric

6192   figures far easier. Such software can enhance the experience of making the

6193   drawings in the constructions described previously. In addition, the ease of making

6194   accurate drawings encourages the formulation and exploration of geometric

6195   conjectures. For example, it is now easy to convince oneself that the intersections of

6196   adjacent angle trisectors of the angles of a triangle are most likely the vertices of an

6197   equilateral triangle (Morley's theorem). If students do have access to such software,

6198   the potential for a more intense mathematical encounter is certainly there. In

6199   encouraging students to use the technology, however, one should not lose sight of

6200   the fact that the excellent visual evidence thus provided must never be taken as a

6201   replacement for understanding. For example, software may give the following

6202   heuristic evidence for why the sum of the angles of a triangle is 180°. When any

6203   three points on the screen are clicked, a triangle with these three points as vertices

6204   appears. When each angle is clicked again, three numbers will appear that give the

6205   angle measurement of each angle. When these numbers are added, 180° will be the

6206   answer. Furthermore, no matter the shape of the triangle, the result will always be

6207   the same.

6208   While such exercises may boost one's belief in the validity of the theorem about

6209   the sum of the angles, it must be recognized that these angle measurements have

6210   added nothing to one's understanding of why this theorem is true. Furthermore, if

6211   one really wants to have a hands-on experience with angle measurements in order

6212   to check the validity of this theorem, the best way is to do it painstakingly by hand on

6213   paper. Morley's theorem, mentioned earlier, is another illustration of the same

6214   principle: evidence cannot replace proofs. The computer program would not reveal

6215   *the reason* the three points are always the vertices of an equilateral triangle.

6216   ***Introduction to the coordinate plane.*** Students should know that the coordinate

6217   plane provides a concrete example that satisfies all the axioms of Euclidean

6218   geometry if the lines are defined as the graphs of linear equations $ax + by = c$, with

6219   at least one of *a* and *b* not equal to zero. Lines $a_1 x + b_1 y = c_1$ and $a_2 x + b_2 y = c_2$

6220   are defined as parallel if (a_1, b_1) is proportional to (a_2, b_2), but (a_1, b_1, c_1) is not

6221   proportional to (a_2, b_2, c_2). The verification of the axioms is straightforward.

6222 **Chapter 3: Algebra II (Grades 8-12)**

6223

6224   Algebra II expands on the mathematical content of Algebra I and geometry. There

6225 is no single unifying theme. Instead, many new concepts and techniques are

6226 introduced that will be basic to more advanced courses in mathematics and the

6227 sciences and useful in the workplace. In general terms the emphasis is on abstract

6228 thinking skills, the function concept, and the algebraic solution of problems in various

6229 content areas.

6230 **Absolute Value and Inequalities**

6231   The study of absolute value and inequalities is extended to include simultaneous

6232 linear systems; it paves the way for linear programming—the maximization or

6233 minimization of linear functions over regions defined by linear inequalities. The

6234 relevant standards are:

6235   **1.0**   Students solve equations and inequalities involving absolute value.

6236   **2.0**   Students solve systems of linear equations and inequalities (in two or three

6237   variables) by substitution, with graphs, or with matrices.

6238   The concept of Gaussian elimination should be introduced for 2 x 2 matrices and

6239 simple 3 x 3 ones. The emphasis is on concreteness rather than on generality.

6240 Concrete applications of simultaneous linear equations and linear programming to

6241 problems in daily life should be brought out, but there is no need to emphasize linear

6242 programming at this stage. While it would be inadvisable to advocate the use of

6243 graphing calculators all the time, such calculators are helpful for graphing regions in

6244 connection with linear programming once the students are past the initial stage of

6245 learning.

6246   **Complex Numbers**

6247   At this point of students' mathematical development, knowledge of complex

6248   numbers is indispensable:

6249   **5.0** Students demonstrate knowledge of how real and complex numbers are

6250   related both arithmetically and graphically. In particular, they can plot

6251   complex numbers as points in the plane.

6252   **6.0** Students add, subtract, multiply, and divide complex numbers.

6253   From the beginning it is important to stress the geometric aspect of complex

6254   numbers; for example, the addition of two complex numbers can be shown in terms

6255   of a parallelogram. And the key difference between real and complex numbers

6256   should be pointed out: The complex numbers cannot be linearly ordered in the same

6257   way as real numbers are (the real *line*).

6258   **Polynomials and Rational Expressions**

6259   The next general technique is the *formal* algebra of polynomials and rational

6260   expressions:

6261   **3.0** Students are adept at operations on polynomials, including long division.

6262   **4.0** Students factor polynomials representing the difference of squares, perfect

6263   square trinomials, and the sum and difference of two cubes.

6264   **7.0** Students add, subtract, multiply, divide, reduce, and evaluate rational

6265   expressions with monomial and polynomial denominators and simplify

6266   complicated rational expressions, including those with negative exponents in

6267   the denominator.

6268   The importance of formal algebra is sometimes misunderstood. The argument

6269   against it is that it has insufficient real-world relevance and it leads easily to an

6270   overemphasis on mechanical drills. There seems also to be an argument for placing

6271   the study of exponential function ahead of polynomials in school mathematics

6272   because exponential functions appear in many real-world situations (compound

6273   interest, for example). There is a need to affirm the primacy of polynomials in high

6274   school mathematics and the importance of formal algebra. The potential for abuse in

6275   Standard 3.0 is all too obvious, but such abuse would be realized only if the

6276   important ideas implicit in it are not brought out. These ideas all center on the

6277   abstraction and hence on the generality of the formal algebraic operations on

6278   polynomials. Thus the division algorithm (long division) leads to the understanding of

6279   the roots and factorization of polynomials. The factor theorem, which states that

6280   ($x-a$) divides a polynomial $p(x)$ if and only if $p(a) = 0$, should be proved; and students

6281   should know the proof. The rational root theorem could be proved too, but only if

6282   there is enough time to explain it carefully; otherwise, many students would be

6283   misled into thinking that *all* the roots of a polynomial with integer coefficients are

6284   determined by the divisibility properties of the first and last coefficients.

6285   It would be natural to first prove the division algorithm and the factor theorem for

6286   polynomials with real coefficients. But it would be vitally important to revisit both and

6287   to point out that the same proofs work, verbatim, for polynomials with *complex*

6288   coefficients. This procedure not only provides a good exercise on complex numbers

6289   but also nicely illustrates the built-in generality of formal algebra.

6290   Two remarks about Standard 7.0 are relevant: (1) a rational expression should be

6291   treated formally, and its function-theoretic aspects (the domain of definition, for

6292   example) need not be emphasized at this juncture; and (2) fractional exponents of

6293   polynomials and rational expressions should be carefully discussed here.

6294 **Quadratic Functions**

6295 The first high point of the course is the study of quadratic (polynomial) functions:

6296 **8.0** Students solve and graph quadratic equations by factoring, completing the
6297 square, or using the quadratic formula. Students apply these techniques in
6298 solving word problems. They also solve quadratic equations in the complex
6299 number system.

6300 **9.0** Students demonstrate and explain the effect that changing a coefficient has
6301 on the graph of quadratic functions; that is, students can determine how the
6302 graph of a parabola changes as a, b, and c vary in the equation $y = a(x-b)^2$
6303 $+ c$.

6304 **10.0** Students graph quadratic functions and determine the maxima, mimima, and
6305 zeros of the function.

6306 What distinguishes Standard 8.0 from the same topic in Algebra I is the newly
6307 acquired generality of the quadratic formula: It now solves all equations $ax^2 + bx + c$
6308 $= 0$ with real a, b, and c regardless of whether or not $b^2 - 4ac < 0$, and it does so
6309 even when a, b, and c are *complex* numbers. Again it should be stressed that the
6310 purely *formal* derivation of the quadratic formula makes it valid for any object a, b,
6311 and c as long as the usual arithmetic operations on numbers can be applied to them.
6312 In particular, it makes no difference whether the numbers are real or complex. This
6313 premise illustrates the built-in generality of formal algebra. Students need to know
6314 every aspect of the proof of the quadratic formula. They should also be made aware
6315 (1) that with the availability of complex numbers, any quadratic polynomial $ax^2 + bx$
6316 $+ c = 0$ with real or complex a, b, and c can be factored into a product of two linear
6317 polynomials with complex coefficients; (2) that c is the product of the roots and $-b$ is
6318 their sum; and (3) that if a, b, and c are real and the roots are complex, then the
6319 roots are a conjugate pair.

6320   Standard 9.0 brings the study of quadratic polynomials to a new level by regarding

6321   them as a function. This new point of view leads to the exact location of the

6322   maximum, minimum, and zeros of this function by use of the quadratic formula (or,

6323   more precisely, by completing the square) without recourse to calculus. The practical

6324   applications of these results are as important as the theory.

6325   Another application of completing the square is given in Standard 17.0, through

6326   which students learn, among other things, how to bring a quadratic polynomial in x

6327   and y without an xy term to standard form and recognize whether it represents an

6328   ellipse or a hyperbola.

6329   **Logarithms**

6330   A second high point of Algebra II is the introduction of two of the basic functions in

6331   all of mathematics: e^x and log x.

6332   **11.0** Students prove simple laws of logarithms.

6333   11.1.   Students understand the inverse relationship between exponents and

6334   logarithms and use this relationship to solve problems involving

6335   logarithms and exponents.

6336   11.2.   Students judge the validity of an argument according to whether the

6337   properties of real numbers, exponents, and logarithms have been

6338   applied correctly at each step.

6339   **12.0** Students know the laws of fractional exponents, understand exponential

6340   functions, and use these functions in problems involving exponential growth

6341   and decay.

6342   **15.0** Students determine whether a specific algebraic statement involving rational

6343   expressions, radical expressions, or logarithmic or exponential functions is

6344   sometimes true, always true, or never true.

6345   The theory should be done carefully, and students are responsible for the proofs

6346   of the laws of exponents for a^m where m is a rational number and of the basic

6347   properties of $\log a\, x$: $\log a\, (x1\ x2) = \log a\, (x1) + \log a\, (x2)$, $\log a\, (1/x) = -\log a\, x$, and

6348 $\log a(x^r) = r\log a\, x$, where r is a rational number (Standard 15.0). The functional

6349   relationships $\log a(a^x) = x$ and $a^{\log(t)} = t$ where a is the base of the log function in the

6350   second equation should be taught without a detailed discussion of inverse functions

6351   in general, as students are probably not ready for it yet. Practical applications of this

6352   topic to growth and decay problems are legion.

6353   **Arithmetic and Geometric Series**

6354   A third high point of Algebra II is the study of arithmetic and geometric series:

6355   **23.0** Students derive the summation formulas for arithmetic series and for both

6356   finite and infinite geometric series.

6357   The geometric series, finite and infinite, is of great importance in mathematics and

6358   the sciences, physical and social. Students should be able to recognize this series

6359   under all its guises and compute its sum with ease. In particular, they should know

6360   by heart the basic identity that underlies the theory of geometric series:

6361 $$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}\, y + \cdots + xy^{n-2} + y^{n-1}).$$

6362   This identity gives another example of the utility of formal algebra, and the identity

6363   is used in many other places as well (the differentiation of monomials, for example).

6364   It should be mentioned that while it is tempting to discuss the arithmetic and

6365   geometric series using the sigma notation

6366 $$\sum_{i=1}^{n},$$

6367   it would be advisable to resist this temptation so that the students are not

6368   overburdened.

6369 **Binomial Theorem**

6370   Students should learn the binomial theorem and how to use it:

6371   **20.0** Students know the binomial theorem and use it to expand binomial

6372   expressions that are raised to positive integer powers.

6373   **18.0** Students use fundamental counting principles to compute combinations and

6374   permutations.

6375   **19.0** Students use combinations and permutations to compute probabilities.

6376   In this context the applications almost come automatically with the theory.

6377   Finally, Standards 16.0 (geometry of conic sections), 24.0 (composition of

6378 functions and inverse functions), and 25.0 may be taken up if time permits.

6379   **Chapter 3: Trigonometry (Grades 8-12)**

6380

6381   Trigonometry uses the techniques that students have previously learned from the

6382   study of algebra and geometry. The trigonometric functions studied are defined

6383   geometrically rather than in terms of algebraic equations, but one of the goals of this

6384   course is to acquaint students with a more algebraic viewpoint toward these

6385   functions.

6386   Students should have a clear understanding that the definition of the trigonometric

6387   functions is made possible by the notion of similarity between triangles.

6388   A basic difficulty confronting students is one of superabundance: There are six

6389   trigonometric functions and seemingly an infinite number of identities relating to

6390   them. The situation is actually very simple, however. Sine and cosine are by far the

6391   most important of the six functions. Students must be thoroughly familiar with their

6392   basic properties, including their graphs and the fact that they give the coordinates of

6393   every point on the unit circle (Standard 2.0). Moreover, three identities stand out

6394   above all others: $\sin^2 x + \cos^2 x = 1$ and the addition formulas of sine and cosine:

6395   **3.0** Students know the identity $\cos^2 (x) + \sin^2 (x) = 1$:

6396   3.1. Students prove that this identity is equivalent to the Pythagorean

6397   theorem (i.e., students can prove this identity by using the Pythagorean

6398   theorem and, conversely, they can prove the Pythagorean theorem

6399   as a consequence of this identity).

6400   3.2. Students prove other trigonometric identities and simplify others by using

6401   the identity $\cos^2 (x) + \sin^2 (x) = 1$. For example, students use this identity

6402   to prove that $\sec^2 (x) = \tan^2 (x) + 1$.

6403   **10.0** Students demonstrate an understanding of the addition formulas for sines

6404   and cosines and their proofs and can use those formulas to prove and/or

6405   simplify other trigonometric identities.

6406   Students should know the proofs of these addition formulas. An acceptable

6407   approach is to use the fact that the distance between two points on the unit circle

6408   depends only on the angle between them. Thus, suppose that angles a and b satisfy

6409 $0 < a < b$, and let A and B be points on the unit circle making angles a and b with the

6410   positive x-axis. Then $A = (\cos a, \sin a)$, $B = (\cos b, \sin b)$, and the distance $d(A, B)$

6411   from A to B satisfies the equation:

6412 $$d(A, B)^2 = (\cos b - \cos a)^2 + (\sin b - \sin a)^2$$

6413   On the other hand, the angle from A to B is $(b - a)$, so that the distance from the

6414   point $C = (\cos (b - a), \sin (b - a))$ to $(1, 0)$ is also $d(A, B)$ because the angle from C

6415   to $(1, 0)$ is $(b - a)$ as well. Thus:

6416 $$d(A, B)^2 = (\cos (b - a) - 1)^2 + \sin^2 (b - a)$$

6417   Equating the two gives the formula:

6418 $$\cos (b - a) = \cos a \cos b + \sin a \sin b$$

6419   From this formula both the sine and cosine addition formulas follow easily.

6420   Students should also know the special cases of these addition formulas in the

6421   form of half-angle and double-angle formulas of sine and cosine (Standard 11.0).

6422   These are important in advanced courses, such as calculus. Moreover, the addition

6423   formulas make possible the rewriting of trigonometric sums of the form $A\sin(x) +$

6424 $B\cos(x)$ as $C\sin(x + D)$ for suitably chosen constants C and D, thereby showing that

6425   such a sum is basically a displaced sine function. This fact should be made known to

6426   students because it is important in the study of wave motions in physics and

6427   engineering.

6428   Students should have a moderate amount of practice in deriving trigonometric

6429   identities, but identity proving is no longer a central topic.

6430   Of the remaining four trigonometric functions, students should make a special

6431   effort to get to know tangent, its domain of definition $(-\pi/2, \pi/2)$, and its graph

6432   (Standard 5.0). The tangent function naturally arises because of the standard:

6433   **7.0** Students know that the tangent of the angle that a line makes with the *x*-axis

6434   is equal to the slope of the line.

6435   Because trigonometric functions arose historically from computational needs in

6436   astronomy, their practical applications should be stressed (Standard 19.0). Among

6437   the most important are:

6438   **13.0** Students know the law of sines and the law of cosines and apply those laws

6439   to solve problems.

6440   **14.0** Students determine the area of a triangle, given one angle and the two

6441   adjacent sides.

6442   These formulas have innumerable practical consequences.

6443   Complex numbers can be expressed in polar forms with the help of trigonometric

6444   functions (Standard 17.0). The geometric interpretations of the multiplication and

6445   division of complex numbers in terms of the angle and modulus should be

6446   emphasized, especially for complex numbers on the unit circle. Mention should be

6447   made of the connection between the *n*th roots of 1 and the vertices of a regular *n*-

6448   gon inscribed in the unit circle:

6449   **18.0** Students know DeMoivre's theorem and can give *n*th roots of a complex

6450   number given in polar form.

6451 **Chapter 3: Mathematical Analysis (Grades 8-12)**

6452

6453   This discipline combines many of the trigonometric, geometric, and algebraic

6454   techniques needed to prepare students for the study of calculus and other advanced

6455   courses. It also brings a measure of closure to some topics first brought up in earlier

6456   courses, such as Algebra II. The functional viewpoint is emphasized in this course.

6457 **Mathematical Induction**

6458   The eight standards are fairly self-explanatory. However, some comments on four

6459   of them may be of value. The first is mathematical induction:

6460   **3.0**   Students can give proofs of various formulas by using the technique of

6461   mathematical induction.

6462   This basic technique was barely hinted at in Algebra II; but at this level, to

6463   understand why the technique works, students should be able to use the technique

6464   fluently and to learn enough about the natural numbers. They should also see

6465   examples of why the step to get the induction started and the induction step itself are

6466   both necessary. Among the applications of the technique, students should be able to

6467   prove by induction the binomial theorem and the formulas for the sum of squares

6468   and cubes of the first n integers.

6469 **Roots of Polynomials**

6470   Roots of polynomials were not studied in depth in Algebra II, and the key theorem

6471   about them was not mentioned:

6472   **4.0**   Students know the statement of, and can apply, the fundamental theorem of

6473   algebra.

6474   This theorem should not be proved here because the most natural proof requires

6475   mathematical techniques well beyond this level. However, there are "elementary"

6476   proofs that can be made accessible to some of the students. In a sense this theorem

6477   justifies the introduction of complex numbers. An application that should be

6478   mentioned and proved on the basis of the fundamental theorem of algebra is that for

6479   polynomials with real coefficients, complex roots come in conjugate pairs.

6480   Consequently, all polynomials with real coefficients can be written as the product of

6481   real quadratic polynomials. The quadratic formula should be reviewed from the

6482   standpoint of this theorem.

6483   **Conic Sections**

6484   The third area is conic sections (see Standard 5.0). Students learn not only the

6485   geometry of conic sections in detail (e.g., major and minor axes, asymptotes, and

6486   foci) but also the *equivalence* of the algebraic and geometric definitions (the latter

6487   refers to the definitions of the ellipse and hyperbola in terms of distances to the foci

6488   and the definition of the parabola in terms of distances to the focus and directrix). A

6489   knowledge of conic sections is important not only in mathematics but also in

6490   classical physics.

6491   **Limits**

6492   Finally, students are introduced to limits:

6493   **8.0**   Students are familiar with the notion of the limit of a sequence and the limit of

6494   a function as the independent variable approaches a number or infinity. They

6495   determine whether certain sequences converge or diverge.

6496   This standard is an introduction to calculus. The discussion should be intuitive and

6497   buttressed by much numerical data. The calculator is useful in helping students

6498   explore convergence and divergence and guess the limit of sequences. If desired,

296

6499   the precise definition of *limit* can be carefully explained; and students may even be

6500   made to memorize it, but it should not be emphasized. For example, students can be

6501   taught to prove why for linear functions $f\ (\ x\ )$ $\lim_{x \to a} f(x) = f(a)$ for any *a*, but it is

6502   more likely a ritual of manipulating ε's and δ's in a special situation than a real

6503   understanding of the concept. The time can probably be better spent on other proofs

6504   (e.g., mathematical induction).

6505	**Chapter 3: Probability and Statistics**
6506	

6507   This discipline is an introduction the study of probability, interpretation of data, and

6508   fundamental statistical problem solving. Mastery of this academic content will

6509   provide students with a solid foundation in probability and facility in processing

6510   statistical information.

6511   Some of the topics addressed review material found in the standards for the

6512   earlier grades and reflect that this content should not disappear from the curriculum.

6513   These topics include the material with respect to measures of central tendency and

6514   the various display methods in common use, as stated in these standards:

6515   **6.0** Students know the definitions of the *mean, median,* and *mode* of a

6516   distribution of data and can compute each in particular situations.

6517   **8.0** Students organize and describe distributions of data by using a number of

6518   different methods, including frequency tables, histograms, standard line and

6519   bar graphs, stem-and-leaf displays, scatterplots, and box-and-whisker plots.

6520   In the early grades students also receive an introduction to probability at a basic

6521   level. The next topic will expand on this base so that students can find probabilities

6522   for multiple discrete events in various combinations and sequences. The standards

6523   in Algebra II related to permutations and combinations and the fundamental counting

6524   principles are also reflective of the content in these standards:

6525   **1.0** Students know the definition of the notion of *independent events* and can use

6526   the rules for addition, multiplication, and complementation to solve for

6527   probabilities of particular events in finite sample spaces.

6528   **2.0** Students know the definition of *conditional probability* and use it to solve for

6529   probabilities in finite sample spaces.

6530   **3. 0** Students demonstrate an understanding of the notion of *discrete random*

6531   *variables* by using them to solve for the probabilities of outcomes, such as

6532   the probability of the occurrence of five heads in 14 coin tosses.

6533   The most substantial new material in this discipline is found in Standard 4.0:

6534   **4.0** Students are familiar with the standard distributions (normal, binomial, and

6535   exponential) and can use them to solve for events in problems in which the

6536   distribution belongs to those families.

6537   Instruction typically flows from the counting principles for discrete binomial

6538   variables to the rules for elaborating probabilities in binomial distributions. The fact

6539   that these probabilities are simply the terms in a binomial expansion provides a

6540   strong link to Algebra II and the binomial theorem. From this base, basic probability

6541   topics can be expanded into the treatment of these standard distributions. In the

6542   binomial case students should now be able to define the probability for a range of

6543   possible outcomes for a set of events based on a single-event probability and thus to

6544   develop better understanding of probability and density functions.

6545   The normal distribution, which is the limiting form of a binomial distribution, is

6546   typically introduced next. Students are not to be expected to integrate this

6547   distribution, but they can answer probability questions based on it by referring to

6548   tabled values. Students need to know that the mean and the standard deviation are

6549   parameters for this distribution. Therefore, it is important to understand variance,

6550   based on averaged squared deviation, as an index of variability and its importance in

6551   normal distributions, as stated in these standards:

6552   **5.0** Students determine the mean and the standard deviation of a normally

6553   distributed random variable.

6554   **7.0** Students compute the variance and the standard deviation of a distribution of

6555   data.

6556   Standard 4.0 also includes exponential distributions with applications, for example,

6557   in lifetime of service and radioactive decay problems. Including this distribution

6558   acquaints students with probability calculations for other types of processes. Here,

6559   students learn that the distribution is defined by a scale parameter, and they learn

6560   simple probability computations based on this parameter.